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TECHNICAL NOTES

Free and mixed convection from slender bodies of revolution in porous media

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INTRODUCTION

THE STUDY of heat transfer from the outer surface of a heated body embedded in a saturated porous medium has important applications in geophysics and engineering. However, most previous studies [1–5] are limited to simple geometry like flat plates or cylinders. As pointed out by Cheng [6], for the specific application of geothermal engineering, the model of hot intrusion by flat plate or cylinder is ideal. For the general case of an axisymmetric body of arbitrary shape, the results are few and they are reported only by Merkin [7] and Nakayama and Koyama [8, 9]. Therefore, it is the purpose of this note to study heat transfer, in the form of free, mixed and forced convection, from a slender body of revolution embedded in a saturated porous medium. It is expected that the results obtained will not only provide useful information for applications but also serve as a complement to the previous studies.

ANALYSIS

Consider a slender body of revolution placed in a saturated porous medium. The body surface is at a variable temperature $T_w(x)$ while the ambient fluid is maintained at a constant temperature T_∞. Having invoked the Boussinesq and boundary-layer approximations, the governing equations based on Darcy's law are given by

$$\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial r}=0 \tag{1}$$

$$\frac{\partial u}{\partial r}=\frac{Kg\beta}{\nu}\frac{\partial T}{\partial r} \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3}$$

where x and r are the axial and radial coordinates with the origin placed at the front stagnation point of the body.

The corresponding boundary conditions are:

at the body surface, $r = R(x)$

$$T = T_w(x) = T_\infty + Ax^\lambda, \quad v = 0; \tag{4}$$

at infinity, $r \to \infty$

$$T = T_\infty,\ u = 0 \qquad \text{for free convection} \tag{5a}$$

$$= U_\infty(x) = Bx^m \quad \text{for mixed convection} \tag{5b}$$

where $R(x)$ prescribes the surface shape of the axisymmetric body.

The system of equations (1)–(3) can be reduced to ordinary differential equations by a similarity transformation.

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Free convection

The suitable similarity variables for the free convection problems under consideration are:

$$\eta = Ra\left(\frac{r}{x}\right)^2 \tag{6}$$

$$\psi = \alpha x f(\eta) \tag{7}$$

and

$$\theta = \frac{T-T_\infty}{T_w-T_\infty}. \tag{8}$$

Setting $\eta = a_{nc}$, where a_{nc} is a constant and is numerically small for a slender body, equation (6) prescribes both shape and size of the body with its surface given by

$$R(x) = \left(\frac{\nu\alpha a_{nc}}{Kg\beta A}\right)^{1/2} x^{(1-\lambda)/2}. \tag{9}$$

For problems of practical interest, the value of $\lambda \leqslant 1$. For example, the body is a cylinder when $\lambda = 1$, a paraboloid when $\lambda = 0$, and a cone when $\lambda = -1$. After transformation, the resulting equations are

$$2f' = \theta \tag{10}$$

$$2\eta\theta''+(2+f)\theta'-\lambda f'\theta = 0 \tag{11}$$

with boundary conditions

$$\eta = a_{nc}, \quad \theta = 1, \quad f+(\lambda-1)a_{nc}f' = 0 \tag{12}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 0. \tag{13}$$

It is clear that solutions for $\lambda = 0$ correspond to a paraboloid with constant temperature, while they correspond to a vertical cylinder with linear temperature distribution for $\lambda = 1$. For the latter case, Minkowycz and Cheng [1] have also reported similarity solutions through a different transformation.

Mixed convection

For mixed convection, the appropriate similarity variables are

$$\eta = Pe\left(\frac{r}{x}\right)^2 \tag{14}$$

$$\psi = \alpha x f(\eta). \tag{15}$$

Equations (2) and (3) are transformed to

$$2f' = 1+\frac{Kg\beta A}{\nu B}x^{(\lambda-m)}\theta \tag{16}$$

$$2\eta\theta''+(2+f)\theta'-\lambda f'\theta = 0. \tag{17}$$

It is apparent that equations (16) and (17) will permit simi-

NOMENCLATURE

a	dimensionless radius of the slender body
A	constant defined in equation (4)
B	constant defined in equation (5b)
f	dimensionless stream functions
g	acceleration due to gravity
K	permeability of the porous medium
m	constant defined in equation (5b)
Nu	Nusselt number, hx/k
Pe	Peclet number, $U_\infty x/\alpha$
r	radial coordinate
r_0	radius of cylinder
R	surface shape of the body
Ra	Rayleigh number, $Kg\beta(T_w - T_\infty)x/v\alpha$
T	temperature
u	velocity component in the x-direction $1/r\,(\partial\psi/\partial r)$
U_∞	free stream velocity
v	velocity component in the r-direction, $-1/r\,(\partial\psi/\partial x)$
x	axial coordinate.

Greek symbols

α	thermal diffusivity of porous medium
β	coefficient of thermal expansion
η	independent similarity variable
$\bar{\eta}$	pseudo-similarity variable, $(Ra^{1/2}/x)[(r_0/2)(r^2/r_0^2)-1]$
θ	dimensionless temperature
λ	constant defined in equation (4)
ν	kinematic viscosity
ξ_0	constant, $2/r_0(\nu\alpha/Kg\beta A)^{1/2}$
ψ	stream function.

Subscripts

mx	mixed convection
nc	natural convection
w	condition at wall
∞	condition at infinity.

larity solutions if the exponent of x in equation (16) vanishes, i.e.

$$m = \lambda. \tag{18}$$

Under this restricted condition, equation (16) can be rewritten as

$$2f' = 1 + \frac{Ra}{Pe}\theta \tag{19}$$

where

$$\frac{Ra}{Pe} = \frac{Kg\beta(T_w - T_\infty)}{\nu U_\infty} = \frac{Kg\beta A}{\nu B} \tag{20}$$

with the boundary conditions

$$\eta = a_{mx}, \quad \theta = 1, \quad f + (m-1)a_{mx}f' = 0 \tag{21}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 1/2. \tag{22}$$

The case of uniform flow over a paraboloid at a constant temperature corresponds to $\lambda = 0$, while $\lambda = 1$ corresponds to accelerating flow past a vertical cylinder with a linear temperature variation along its axis.

As is the case for free convection, by setting $\eta = a_{mx}$, equation (14) prescribes both the shape and size of the body with its surface given by

$$R(x) = \left(\frac{a_{mx}}{Pe}\right)^{1/2} x. \tag{23}$$

For a given body, the relation between a_{nc} and a_{mx} is given by

$$a_{nc} = \frac{Ra}{Pe} a_{mx}. \tag{24}$$

Forced convection

For the limiting case of forced convection, it is noted that the governing equations can be readily derived from equations (17) and (19) by simply setting $Ra/Pe = 0$. Therefore

$$f' = 1/2 \tag{25}$$

$$2\eta\theta'' + [2 + (\eta - ma_{mx})/2]\theta' - \lambda\theta/2 = 0. \tag{26}$$

RESULTS AND DISCUSSION

The transformed sets of ordinary differential equations, with their corresponding boundary conditions, are solved by numerical integration using the fourth order Runge–Kutta method and the shooting technique with a systematic guessing of $\theta'(a)$.

The heat transfer coefficient in terms of the Nusselt number is given by

$$\frac{Nu}{Ra^{1/2}} = [-2a^{1/2}\theta'(a)]_{nc}, \quad \text{for free convection} \tag{27}$$

$$\frac{Nu}{Pe^{1/2}} = [-2a^{1/2}\theta'(a)]_{mx}, \quad \text{for mixed and forced convection.} \tag{28}$$

For free convection, similarity solutions have also been reported by Minkowycz and Cheng [1] for a vertical cylinder. Since the transformations they used are different from those of the present study, additional transformations are required for the direct comparison of the heat transfer results. The transformations for such purpose are given by

$$\theta_\eta = \left(\frac{\xi_0}{4}\right)\theta_{\bar{\eta}} \tag{29}$$

where the subscript denotes the differentiation, and $\bar{\eta}$ and ξ_0 are the dimensionless variables employed by Minkowycz and Cheng [1]. In terms of the similarity variable used in the present analysis, they are given by

$$\bar{\eta} = \frac{\xi_0}{4}(\eta - a_{nc}) \tag{30}$$

$$\xi_0 = 2/a_{nc}^{1/2}. \tag{31}$$

The results thus obtained are plotted in Fig. 1 where a nice agreement is clearly observed.

The heat transfer results for natural convection are shown in Fig. 2. It is observed that the heat transfer coefficient decreases with an increase in the dimensionless radius. This trend has also been reported by Minkowycz and Cheng [1].

For mixed convection, the heat transfer result, equation (28), is plotted in Fig. 3 as a function of a_{mx} and Ra/Pe. The limiting cases of free and forced convection are also shown as asymptotes in the same figure. It is interesting to observe that for $\lambda = 1$, the heat transfer coefficient decreases with an increase in the dimensionless radius, while for $\lambda = 0$, the foregoing statement is valid only at a small value of Ra/Pe (<30). The heat transfer coefficient then increases with the dimensionless radius at a higher value of Ra/Pe.

The corresponding free convection asymptotes can be

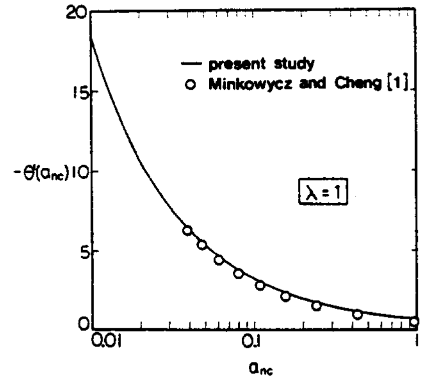

FIG. 1. A comparison of results obtained by different transformations.

obtained by rewriting equation (28) as

$$\frac{Nu}{Pe^{1/2}} = \frac{Nu}{Ra^{1/2}}\left(\frac{Ra}{Pe}\right)^{1/2} = \left(\frac{Ra}{Pe}\right)^{1/2}[-2a^{1/2}\theta'(a)]_{nc} \quad (32)$$

and applying the relation between a_{nc} and a_{mx} as given by equation (24).

With a given a_{mx} and Ra/Pe, a_{nc} can be determined through equation (24). Once a_{nc} is specified, $\theta'(a)$ can be obtained from solutions of equations (10) and (11). Therefore, the free convection asymptote is obtained, from equation (32), for each corresponding a_{mx}.

To summarize, heat transfer in the form of free, mixed and forced convection from a slender body of revolution embedded in a saturated porous medium has been studied analytically. Similarity solutions have been reported for the special cases for which the wall temperature and the free stream velocity are prescribed power functions of distance. Except in the case of $\lambda = 0$ and mixed convection at higher values of Ra/Pe (>30), it is found that the heat transfer coefficient decreases with the dimensionless radius. Problems of this kind may be encountered in geophysical and geothermal applications. The results thus obtained will be helpful in the assessment and evaluation of geothermal energy resources. In addition, the heat transfer coefficient obtained from this study provides useful information to estimate the cooling rate of intrusive bodies and consequently the life span of a geothermal reservoir.

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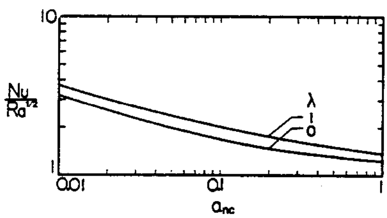

FIG. 2. Heat transfer results for free convection.

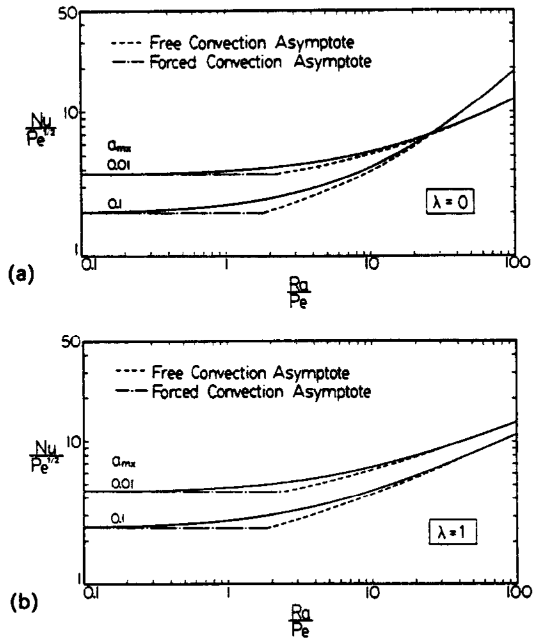

FIG. 3. Heat transfer results for mixed convection. (a) $\lambda = 0$; (b) $\lambda = 1$.

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